Linear Algebra →
- Determinants
- Inverse
- Rank
- Solution of System of Linear equations.
- Eigen values, Eigen Vectors, Cayley-Hamilton theorem.

① Determinants:-

→ $||\bar{x}||$ → norm of $\bar{x}$

→ Determinants is for Square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ → arrangement is called Matrix.

② 2^{nd} Order determinant

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the expression $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called 2^{nd} Order determinant of $A_{2\times2}$ & it is denoted by $|A|$ or $\det(A)$.

The expansion of $|A| = (a_{11}a_{22} - a_{12}a_{21})$

3. 3rd Order determinant :-

$$|A| = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Method 2 :-

$$\begin{array}{ccccc} & 4 & 5 & 6 & \\ a_{13} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & a_{11} \\ \cancel{a_{33}} & & a_{31} \\ & 1 & 2 & 3 & \end{array}$$

$$|A| = \left[\{1 + 2 + 3\} - \{4 + 5 + 6\}\right]$$

4. 4th Order Determinant :-

$$|A| = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \end{array} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

(5) Elementary Operations:-

(i) $R_i \leftrightarrow R_j$

(ii) $R_i \longrightarrow KR_i \quad (K \neq 0)$

(iii) $R_j \longrightarrow R_j + KR_i$

Note: (1) Always select that Row or Column which has more No. of zero to evaluate the value.

(2) Total no. of terms in the expansion of a determinant of order n is $n!$

$$\begin{cases} 2\times 2 \rightarrow 2 = 2! \\ 3\times 3 \rightarrow 6 = 3! \\ 4\times 4 \rightarrow \cancel{24} \quad 4! = 24. \end{cases}$$

(3) If A is $(m\times n)$ and B is $(n\times p)$, then how many no. of multiplication & addition are involved in computing matrix product $A\times B$?

$$[A_{m\times n}] \times [B]_{n\times p} = [AB]_{m\times p}$$

equal always for product of Matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ \vdots & & & & \\ b_{m1} & b_{m2} & \cdots & & b_{mp} \end{bmatrix}_{n\times p}$$

$$= \underbrace{(a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{m1})}_{\substack{(n-1)\text{ additions for 1 term.} \\ n \rightarrow \text{multiplication for 1 terms}}} + (a_{11}b_{12} + a_{22}b_{22} + \cdots + a_{1n}b_{m2}) + \cdots$$

Addition:- $mp(n-1)$ Multiplication (mpn)

Properties of determinants :-

① $|A_{n\times n} B_{n\times n}| = |A_{n\times n}|\,|B_{n\times n}|$

② $|A^K| = |A|^K$

③ $|A^T| = |A|$ (Transpose).

④ If two rows are same, $|A| = 0$

⑤ If two rows are interchanged, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$

$|B| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_{31} & c_{32} & c_{33} \\ b_{21} & b_{22} & b_{23} \end{vmatrix} \overset{(-1)}{=} -|A|$

or $(-1)^K |A|$

$|C| = \begin{vmatrix} c_{31} & c_{32} & c_{33} \\ b_{21} & b_{22} & b_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} \overset{(-1)(-1)}{=} |A|$

K = no. of times of interchange of two rows.

⑥ $\begin{vmatrix} 1 & 2 & 3 \\ 4K & 5K & 6K \\ 7 & 8 & 9 \end{vmatrix} = K \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ → for determinants

for Matrix,

$K \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} K & 2K & 3K \\ 4K & 5K & 6K \\ 7K & 8K & 9K \end{bmatrix}$ → for Matrix.

$|K A_{n\times n}| = K^n |A_{n\times n}|$

(7) $A = \begin{bmatrix} 2 & 0 & 8 \\ 0 & 9 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

→ Upper Traingular Matrix
→ Lower Triangular Matrix
→ Diagonal Matrix.
→ Scalar matrix. = K (identity Matrix).
→ Identity matrix.
→ Null matrix.

(*) Singular Matrix → $|A| = 0$

(*) Idempotent Matrix ; $(A^2 = A)$

(*) Involutory Matrix ; $(A^2 = I)$

Inverse of Square Matrix:-

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = \frac{(\text{Cofactor matrix of } A)^T}{|A|}$$ (Explained Afterwards).

(*) Equality of Matrices:- ~~(Addi~~

$$\begin{bmatrix} x-y & p+q \\ p-q & x+y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

then, $\left.\begin{aligned} x-y &= 2 \\ p+q &= 5 \\ p-q &= 1 \\ x+y &= 10 \end{aligned}\right\}$

(*) Addition / Subtraction of Matrices:-

$$A \pm B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \pm \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{cases} \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix} & \text{(Addition)} \\ \begin{bmatrix} -3 & -4 \\ -4 & -3 \end{bmatrix} & \text{(subtraction)} \end{cases}$$

Note:- (1) Properties of Matrix Multiplication:-

(a) $(AB)C = A(BC)$

(b) $AB = 0 \Rightarrow A \neq 0$ or $B \neq 0$ {not necessary}.

(c) $AB \neq BA$.

(d) $AB = AC \Rightarrow B = C$ (iff A is non-singular Matrix)