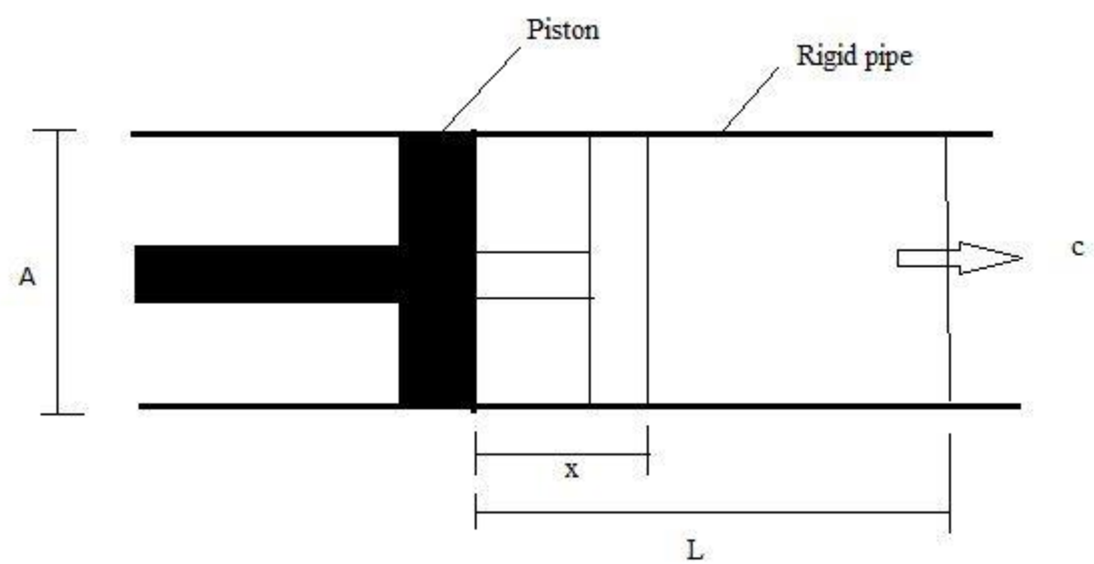
GATE Aerospace Coaching by Team IGC

Compressible Fluid Flow Basics

Velocity of sound in a fluid



A = Cross section area of pipe

V = Velocity of piston

p = pressure of fluid in pipe before movement of piston

ρ = density of fluid before the moment of the piston

dt = small interval of time with which piston is moved

c = Velocity of pressure wave travelling in fluid

Mass of fluid for a length 'L' before compression

$= \rho \times A \times L$

$= \rho \times A \times c \times dt$

Mass of fluid after compression for length (L-x)

= (ρ+dρ) x A x (L-x)

= (ρ+dρ) x A x (cdt-vdt)

From continuity

Mass of fluid before compression = mass of fluid after compression

ρAcdt = (ρ+dρ)A(c-v)dt

cdρ = ρv ………………………………(1) (neglecting vdρ)

net force on fluid element

(p+dp)A-p x A = mass per second x (change in velocity)

Dp x A = $\frac{\rho AL}{dt}[v-o] = \frac{\rho Acdt}{dt}$ x v

c = $\frac{dp}{\rho v}$ …………………………………(2)

multiplying (1) and (2)

$c^2 d\rho = dp$

c = $\sqrt{\frac{dp}{d\rho}}$

Sonic velocity for an adiabatic process

For adiabatic process $\frac{p}{\rho^\gamma}$ = c

diff. above eq.

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho}$$

Here,

$$c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$$

for isothermal process

$$\frac{p}{\rho} = \text{constant}$$

diff. above equation

$$\frac{dp}{d\rho} = \frac{p}{\rho} = \text{RT}$$

Hence,

$$c = \sqrt{RT}$$

Important points about sonic velocity

(1). Sonic velocity is depends upon the change in density for a given change in pressure.

(2). If increase with growth in temperature

(3). Sonic velocity is higher in gases with a high value of gas constant (R)

Mach number (M):-

Define as square root of the ratio of inertia force of a flowing fluid to the elastic force.

$$M = \sqrt{\frac{\rho A v^2}{kA}} = \frac{v}{c}$$

Here,

v = velocity of fluid

c = velocity of sound in the fluid

k = bulk modulus

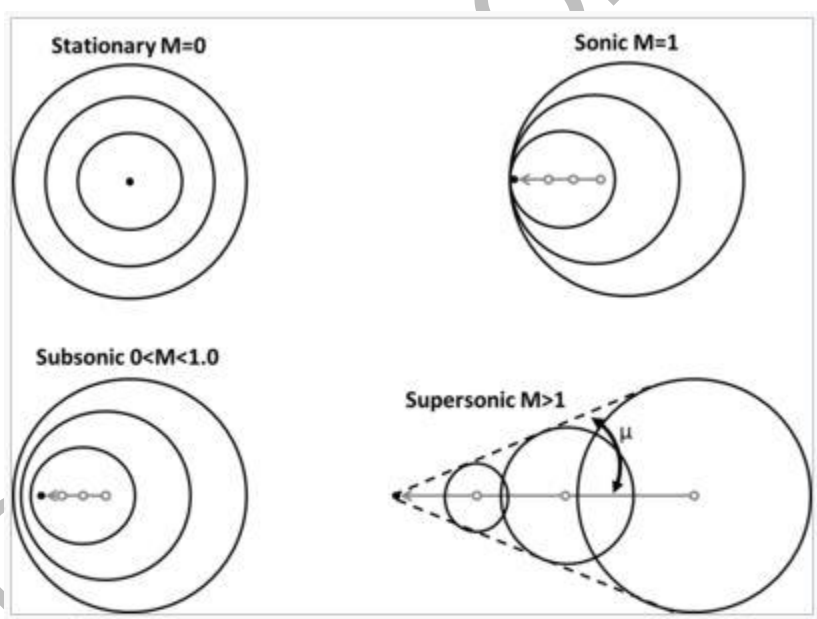
c = $\sqrt{\dfrac{k}{\rho}}$

M < 1 $\rightarrow$ subsonic flow

M = 1 $\rightarrow$ sonic flow

M > 1 $\rightarrow$ supersonic flow

Mach Angle:-



Propagation of disturbance wave

(a),(b) the disturbance wave reach a stationary observer before the source of disturbance could reach him in subsonic flow

$$\sin\mu = \frac{at}{vt} = \frac{a}{v} = \frac{1}{M}$$

$\mu = \sin^{-1}\left(\frac{1}{M}\right)$ (Mach angle)

Compressible flow

Basic equations

(1). Equation of state

pv = mRT

where,

p = absolute pressure in N/m^2

v = volume occupied by mass (m) of the gas

ρ = mass density in kg/m^3

T = absolute temperature in kelvin (K)

R = gas constant (287 J/kg-K)

(2). Continuity equation

ρAv = constant (for 1D steady flow)

differential form

$$\frac{dv}{v} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0$$

(3). Momentum equation (Euler's equation)

$$\frac{dp}{\rho} + vdv + gdz = 0$$

(4). Energy equation

Incompressible flow

Compressible flow

Energy equation (Bernoulli's equ) for incompressible flow

$$\frac{dp}{\rho} + vdv + gdz = 0$$

Integrating above equ

$$\int \frac{dp}{\rho} + \int vdv + \int gdz = \text{constant}$$

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant}$$

For compressible flow

For isothermal process $\Rightarrow \frac{p}{\rho} = \text{constant} = c_1$

$$\rho = \frac{p}{c_1}$$

$$\int \frac{dp}{\rho} = \int \frac{dp}{p} c_1 = c_1 \int \frac{dp}{p} = c_1 \ln p = \frac{p}{\rho} \ln p$$

$$\int \frac{dp}{\rho} + \int vdv + \int gdz = \text{constant}$$

$$\frac{p}{\rho} \ln p + \frac{v^2}{2} + gz = \text{constant}$$

Bernoulli's Equation for adiabatic process ($pv^{\gamma} = c$)

$$\frac{p}{\rho^{\gamma}} = c_1 \Rightarrow \rho = \left(\frac{p}{c_1}\right)^{\frac{1}{\gamma}}$$

$$\int \frac{dp}{\rho} = \int \frac{dp}{p^{\frac{1}{\gamma}}} c_1^{\frac{1}{\gamma}}$$

$$= c_1^{\frac{1}{\gamma}} \left(\frac{p^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} \right)$$

$$= \left(\frac{\gamma}{\gamma-1}\right) \frac{c_1^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}} \text{p}$$

$$= \left(\frac{\gamma}{\gamma-1}\right) \frac{p}{\rho}$$

Hence,

Substituting $\int \frac{dp}{\rho}$ into Euler's momentum equation

$$\int \frac{dp}{\rho} + \int v dv + \int g dz = \text{constant}$$

$$\left(\frac{\gamma}{\gamma-1}\right) \frac{p}{\rho} + \frac{v^2}{2} + gz = \text{constant}$$

Further,

$$\left(\frac{\gamma}{\gamma-1}\right) \frac{p_1}{\rho_1} + \frac{v^2}{2} + gz_1 = \left(\frac{\gamma}{\gamma-1}\right) \frac{p_2}{\rho_2} + \frac{v^2}{2} + gz_2$$

$$\left(\frac{\gamma}{\gamma-1}\right)\left\{\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right\}+\frac{v_1^{\;2}-v_2^{\;2}}{2}+g(z_1-z_2)=0$$

Use,

$$\frac{T_2}{T_1}=\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \quad ; \frac{p_1}{\rho_1}=RT_1 \quad ; c_p=\frac{\gamma R}{\gamma-1} \quad ; \frac{p_1}{\rho_1\gamma}=\frac{p_2}{\rho_2\gamma}$$

Above equation can be reduced to

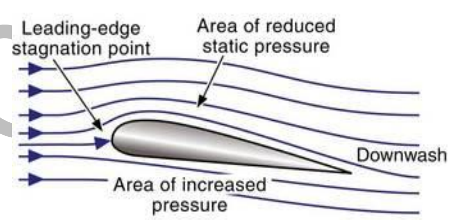
$$c_pT_1+\frac{v_2^{\;2}}{2}+gz_2=c_pT_1+\frac{v_1^{\;2}}{2}+gz_1=\text{constant}$$

Steady flow energy equation

No heat exchange

No shaft work

Stagnation point / stagnation properties



(1) p_o, T_o, ρ_o

v_o (at stagnation point)

using above equation with $z_1=z_2$ at point (1) and (2)

$$c_p T + \frac{v^2}{2} = c_p T_o + \frac{{v_o}^2}{2}$$

$$h + \frac{v^2}{2} = h_o \text{ (Total specific enthalpy)}$$

$$c_p T + \frac{v^2}{2} = c_p T_o$$

$$1 + \frac{1}{2}\frac{v^2}{c_p T} = \frac{T_o}{T} \quad \left\{ c_p = \frac{\gamma R}{\gamma - 1} \right\}$$

$$\frac{T_o}{T} = 1 + \frac{1}{2}\frac{v^2}{T}\frac{\gamma - 1}{\gamma R} \Rightarrow \frac{T_o}{T} = 1 + \frac{1}{2}(\gamma - 1)\frac{v^2}{a^2} \qquad \left\{ a = \sqrt{\gamma RT} \right\}$$

$$\Rightarrow \frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2$$

From adiabatic relation $p_o {v_o}^{\gamma} = p v^{\gamma}$ or $\frac{p_o}{{\rho_o}^{\gamma}} = \frac{p}{\rho^{\gamma}}$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma - 1}} \Rightarrow \frac{p_o}{p} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma - 1}}$$

Similarly,

$$\frac{\rho_o}{\rho} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{1}{\gamma - 1}}$$

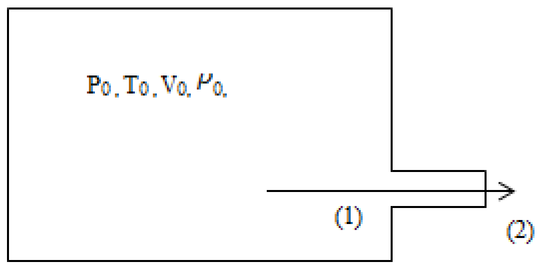
Relation between a and a_o

$$c_p T + \frac{v^2}{2} = c_p T_o$$

$$\frac{\gamma RT}{\gamma - 1} + \frac{v^2}{2} = \frac{\gamma RT_o}{\gamma - 1}$$

$$\frac{a^2}{\gamma-1}+\frac{v^2}{2}=\frac{a_0{}^2}{\gamma-1}$$

Flow of compressible fluid from a reservoir



Apply Bernoulli's equ at (1) and (2) (assuming adiabatic process)

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}+\frac{v_0{}^2}{2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2}+\frac{v_2{}^2}{2}$$

$$v_2=\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}\left(1-\frac{p_2}{\rho_2}\frac{\rho_0}{p_0}\right)}$$

$$\frac{\rho_0}{\rho_2}=\left(\frac{p_0}{p_2}\right)^{\frac{1}{\gamma}}$$

Hence,

$$v_2=\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}\left[1-\left(\frac{p_2}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

V_2 will be maximum when $p_2=0$ {for given p_0, T_0, ρ_0}

$$v_2=\sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}}=a_0\sqrt{\frac{2}{\gamma-1}}=V_{max}$$

$$v_2 = \sqrt{2c_p T_0}$$