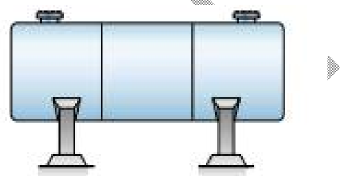
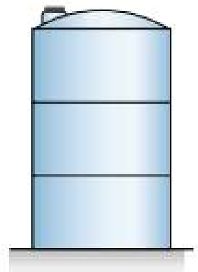
Thin Shell (Cylindrical and Spherical Shell)

1. Thin Cylinder

Closed vessels are used for storing fluids under pressure. If the ratio of thickness of shell to internal radius is less than 1/10, then the cylindrical vessel is known as thin cylinder. In this cylinder, distribution of stress is assumed to be uniform over the thickness of wall.



(a)

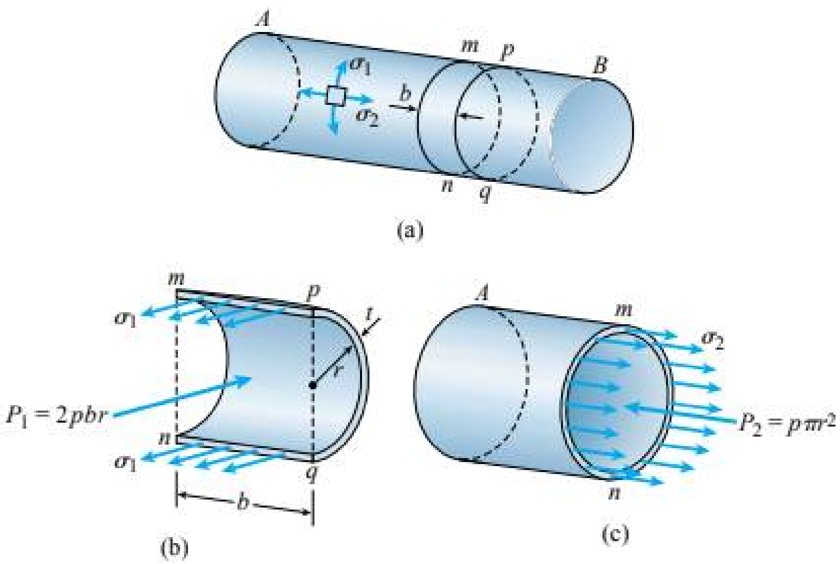


(ref Gere)

1.1 Stresses in a thin cylindrical shell

When a thin cylinder is subjected to internal pressure, its wall is subjected to two types of tensile stresses.

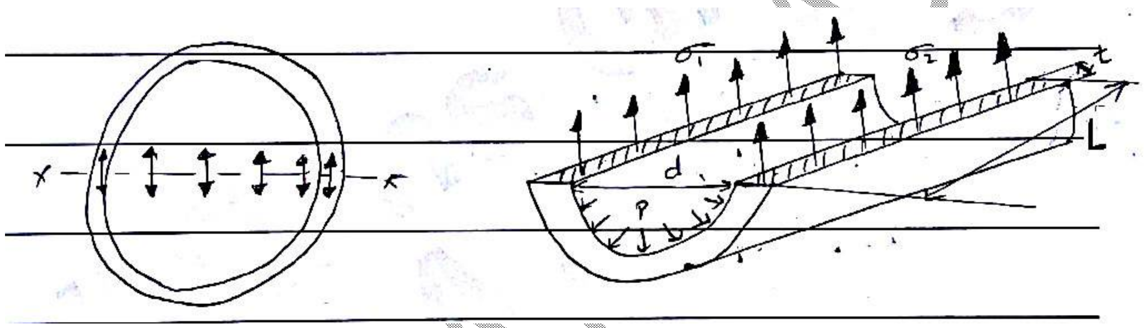
I. Circumferential stress or Hoop stress
II. Longitudinal stress



(ref Gere)

- **Circumferential stress**

Consider a thin cylindrical shell subjected to an internal pressure as shown below. Due to circumferential stress, the cylinder has a tendency to split up into two parts.



σ_1 or f_1 or σ_c = circumferential hoop stress.

The bursting will take place if the force due to fluid pressure is more than the resisting force due to the circumferential stress.

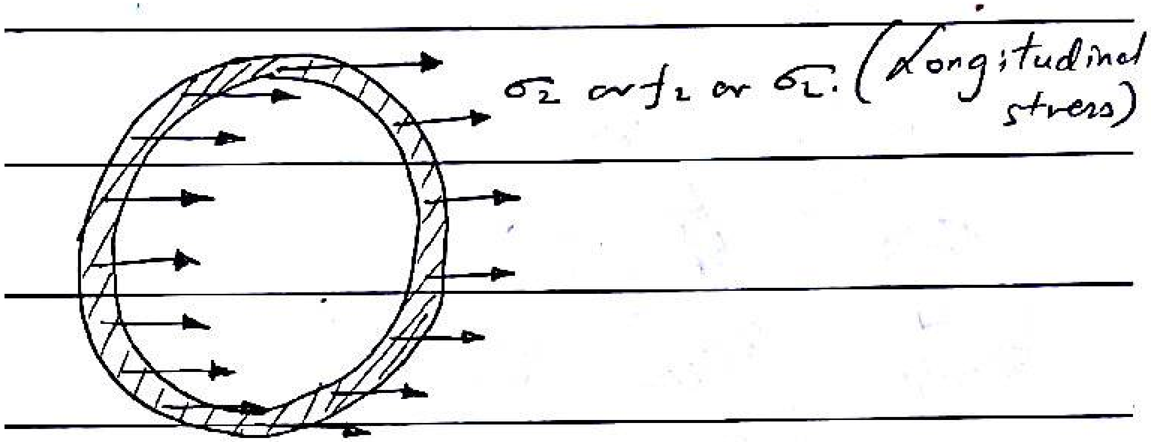
Therefore from force equilibrium,

$2tl\sigma_c = pdl$

$\sigma_c = \frac{Pd}{2t}$ This stress is tensile.

- **Longitudinal stress**

Consider a thin cylinder subjected to an internal pressure as shown below.

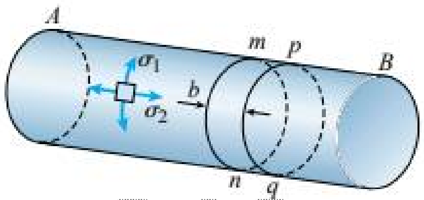


From force equilibrium,

$\sigma_L \times \pi dt = P \times \frac{\pi}{4} \times d^2$

$\boldsymbol{\sigma_L} = \frac{\boldsymbol{Pd}}{\boldsymbol{4t}}$ (tensile)

- **Principal stresses**



(ref Gere)

Here $\boldsymbol{\sigma_c}$ **and** $\boldsymbol{\sigma_L}$ **are principal stresses as there is no shear stress on stress element**

Max principal stress is $\boldsymbol{\sigma_c} = \frac{\mathbf{Pd}}{\mathbf{2t}}$

- **In plane Maximum shear stress**

At any point in a material of thin cylindrical shell, there are two principal stresses – circumferential stress and longitudinal stress. These two stresses are tensile and perpendicular to each other. The element is subjected to maximum shear stress on a plane which at an angle of 45° with respect to longitudinal axis.

Maximum shear stress,

$$\tau_{max3} = \frac{\sigma_C - \sigma_L}{2} \text{ (in-plane)}$$

$$= \left(\frac{\mathbf{Pd}}{\mathbf{2t}} - \frac{\mathbf{Pd}}{\mathbf{4t}}\right)/\mathbf{2}$$

$$= \frac{\mathbf{Pd}}{\mathbf{8t}} \text{ (in-plane)}$$

- **Out of plane Maximum shear stress**

$$\tau_{max1} = \frac{\sigma_C - 0}{2}$$

$$\tau_{max1} = \frac{\sigma_C}{2}$$

$$\tau_{max1} = \frac{\mathbf{Pd}}{\mathbf{4t}} \text{ (Out of-plane)}$$

$$\tau_{max2} = \frac{\sigma_l - 0}{2}$$

$$\tau_{max1=}\ \frac{\sigma_l}{2}$$

$$\tau_{max1=}\ \frac{\mathbf{Pd}}{\mathbf{8t}}\text{(Out of-plane)}$$

Note in theory of failure for thin cylindrical shell using tresca's theory τ_{max1} must be used as it is max among all the max shear stresses.

1.2 Change in dimension of thin cylinder

- **Circumferential strain and change in diameter**

By generalized Hook's law,

Circumferential strain $\epsilon_c = \frac{\sigma_c}{E} - \nu\frac{\sigma_L}{E}$

$$= \frac{Pd}{2tE} - \nu\frac{Pd}{4tE}$$

$$\epsilon_c = \frac{Pd}{2tE}\left(1 - \frac{\nu}{2}\right) = \frac{\delta d}{d}$$

$\delta d = \frac{Pd^2}{2tE}(1 - \frac{\nu}{2})$ (change in diameter)

- **Longitudinal strain and change in length**

Longitudinal strain $\epsilon_l = \frac{\sigma_L}{E} - \nu \frac{\sigma_C}{E}$

$= \frac{Pd}{4tE} - \nu \frac{Pd}{2tE}$

$\epsilon_l = \frac{Pd}{4tE}(1 - 2\nu) = \frac{\delta l}{l}$

$\delta l = \frac{Pdl}{4tE}(1 - 2\nu)$ (change in length)

- **Volumetric strain and change in volume**

Original volume of thin shell = $\frac{\pi d^2 l}{4}$

Differentiating → $\delta V = \frac{\pi}{4} \times 2d \times \delta d \times l + \frac{\pi d^2 \times \delta l}{4}$

Volumetric strain, $\epsilon_V = \frac{\delta V}{V} = \frac{\frac{\pi}{4}\times 2d\times\delta d\times l + \frac{\pi d^2\times\delta l}{4}}{\frac{\pi d^2 l}{4}}$

$\frac{\delta V}{V} = \frac{2\delta d}{d} + \frac{\delta l}{l}$

$\epsilon_V = 2\epsilon_1 + \epsilon_2$

Change in volume = $\delta V = (2\epsilon_1 + \epsilon_2).V$

$$\delta V = \frac{Pdl}{4tE}(5 - 4\nu).V$$

Change in thickness of cylindrical shell

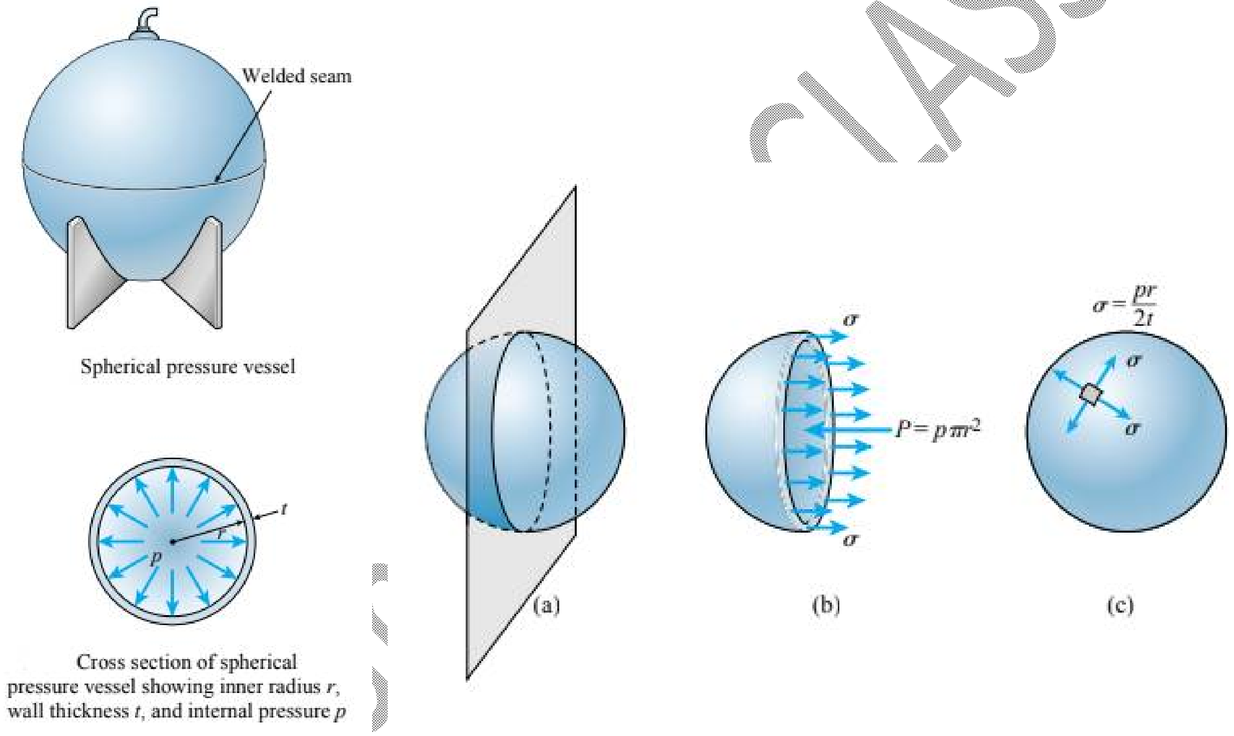
Strain along radial direction from Poisson effect = $-\nu\left(\frac{\sigma_L}{E} + \frac{\sigma_C}{E}\right)$

$$\frac{\delta t}{t} = -\nu\frac{3Pd}{4tE}$$

$$\delta t = -\nu\frac{3Pd}{4E}$$

2. Thin spherical shell

The fluid inside the shell has the tending to split the shell into two hemispheres.



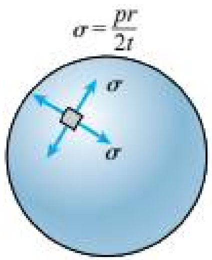
(ref Gere)

From force equilibrium,

$$\sigma \times \pi dt = P \times \frac{\pi}{4} d^2$$

$$\boldsymbol{\sigma = \frac{Pd}{4t} = \sigma_c = \sigma_L}$$

- **Principal stresses**



(ref Gere)

Here **all the directions are principal directions**

Max principal stress is $\boldsymbol{\sigma_1} = \frac{\mathbf{Pd}}{\mathbf{4t}}$

- **In plane Maximum shear stress**

At any point in a material of thin spherical shell, there are two principal stresses are same in all orientations. These two stresses are tensile and perpendicular to each other. For thin spherical in-plane max shear stress is zero and all planes are principal planes

Maximum shear stress,

$$\tau_{max3} = \frac{\sigma_C - \sigma_L}{2} \text{ (in-plane)}$$

$$\tau_{max3} = 0$$

- **Out of plane Maximum shear stress**

$$\tau_{max1} = \frac{\sigma_C - 0}{2}$$

$$\tau_{max1} = \frac{\sigma_C}{2}$$

$$\tau_{max1} = \frac{\mathbf{Pd}}{\mathbf{8t}} \text{ (Out of-plane)}$$

$$\tau_{max2} = \frac{\sigma_l - 0}{2}$$

$\tau_{max2=}\ \frac{\sigma_l}{2}$

$\tau_{max2=}\ \frac{\mathbf{Pd}}{\mathbf{8t}}$(Out of-plane)

Note in theory of failure for thin spherical shell using tresca's theory τ_{max1} must be used as it is max among all the max shear stresses.

- **Change in dimension of a thin spherical shells**

$$\sigma_1 = \sigma_2 = \frac{Pd}{4t}$$

∴ Strain in any direction = $\epsilon = \frac{\sigma_1}{E}$ - $\nu\frac{\sigma_2}{E}$

$$= \frac{\sigma_1}{E}(1-\nu)$$

$$\epsilon = \frac{Pd}{4tE}(1-\nu) = \frac{\delta d}{d}$$

$\therefore\ \delta d = \frac{Pd^2}{4tE}(1-\nu)$

Volume of sphere, V = $\frac{\pi d^3}{6}$

Differentiating → $\delta V = \frac{\pi 3 d^2 . \delta d}{6}$

∴ Volume strain, $\epsilon_V = \frac{\delta V}{V} = \frac{\frac{\pi 3 d^2.\delta d}{6}}{\frac{\pi d^3}{6}} = \frac{3\delta d}{d} = 3\epsilon = \frac{3Pd}{4tE}(1-\nu)$

Change in thickness of spherical shell

Strain along radial direction from Poisson effect = $-\nu\left(\frac{\sigma_L}{E} + \frac{\sigma_C}{E}\right)$

$$\frac{\delta t}{t} = -\nu\frac{Pd}{2tE}$$

$$\delta t = -\nu\frac{Pd}{2E}$$

Problems

1.

A thin cylindrical pressure vessel with closed-ends is subjected to internal pressure. The ratio of circumferential (hoop) stress to the longitudinal stress is

(A) 0.25 (B) 0.50 (C) 1.0 (D) 2.0

2.

A gas is stored in a cylindrical tank of inner radius 7 m and wall thickness 50 mm. The gage pressure of the gas is 2 MPa. The maximum shear stress (in MPa) in the wall is

(A) 35 (B) 70 (C) 140 (D) 280

3.

A thin gas cylinder with an internal radius of 100 mm is subject to an internal pressure of 10 MPa. The maximum permissible working stress is restricted to 100 MPa. The minimum cylinder wall thickness (in mm) for safe design must be

4.

A long thin walled cylindrical shell, closed at both the ends, is subjected to an internal pressure. The ratio of the hoop stress (circumferential stress) to longitudinal stress developed in the shell is

(A) 0.5 (B) 1.0 (C) 2.0 (D) 4.0

5.

A thin cylinder of inner radius 500 mm and thickness 10 mm is subjected to an internal pressure of 5 MPa. The average circumferential (hoop) stress in MPa is

(A) 100 (B) 250 (C) 500 (D) 1000

6.

A closed thin cylindrical pressure vessel having an internal diameter of 1000 mm and a thickness of 10 mm is subjected to an internal pressure of 4 MPa. The maximum shear stress (in MPa) induced in the cylinder is ___________ (neglect the radial stress).

7.

A compressed air tank having an inner diameter of 480 mm and a wall thickness of 8 mm is formed by welding two steel hemispheres. If the allowable shear stress in the steel is 40 MPa, find the maximum permissible pressure (in MPa) inside the tank.

8.

A strain gauge is mounted on the outer surface of a thin cylindrical pressure vessel in the circumferential direction. The mean diameter and thickness of the cylinder are 4.0 m and 20 mm, respectively. Young's modulus and Poisson's ratio of the material of the cylinder are 200 GPa and 0.25, respectively. Find the pressure in MPa inside the cylindrical vessel when the strain gauge indicates a strain of 7.0×10^{-4}. ____________

9.

A thin walled spherical pressure vessel made of a linear elastic isotropic material has inner radius r and thickness t before pressurization. When subjected to internal pressure p, elements of the pressure vessel wall experience a state of stress described by a single point $(\sigma, \tau) = (pr/(2t), 0)$ in Mohr's circle. The reduction of the wall thickness due to pressurization

(A) increases with t
(B) remains independent of t
(C) decreases with t
(D) depends on the elastic properties.