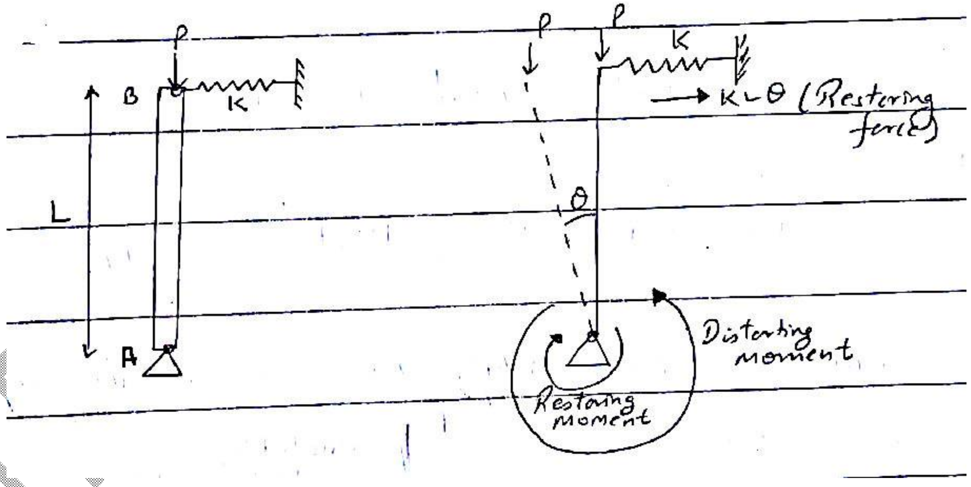
BUCKLING OF COLUMNS

➢ Stability of columns

A prismatic column can be idealized to a spring rigid bar system

Consider an identical structure as shown below. Member AB is a rigid bar that is pinned at base and supported by an elastic spring of stiffness k at top.

The bar supports a centrally applied load P that is perfectly aligned with the axis of bar, hence the spring has to be inserted force in it.



If the bar is disturbed by some external force so that it rotates slightly through a small angle θ about A.

The stability conditions for the column are:

(1) If restoring moment is greater than distorting moment then column is in stable equilibrium.

$$KL^2\theta > PL\,\theta$$

$$KL > P \quad \text{(condition for stable equilibrium)}$$

(2) If restoring moment is equal to distorting moment then column is in neutral equilibrium.

$$KL^2\theta = PL\,\theta$$

$$KL = P \quad \text{(condition for neutral equilibrium)}$$

Here the load P is known as a critical load P_{cr}.

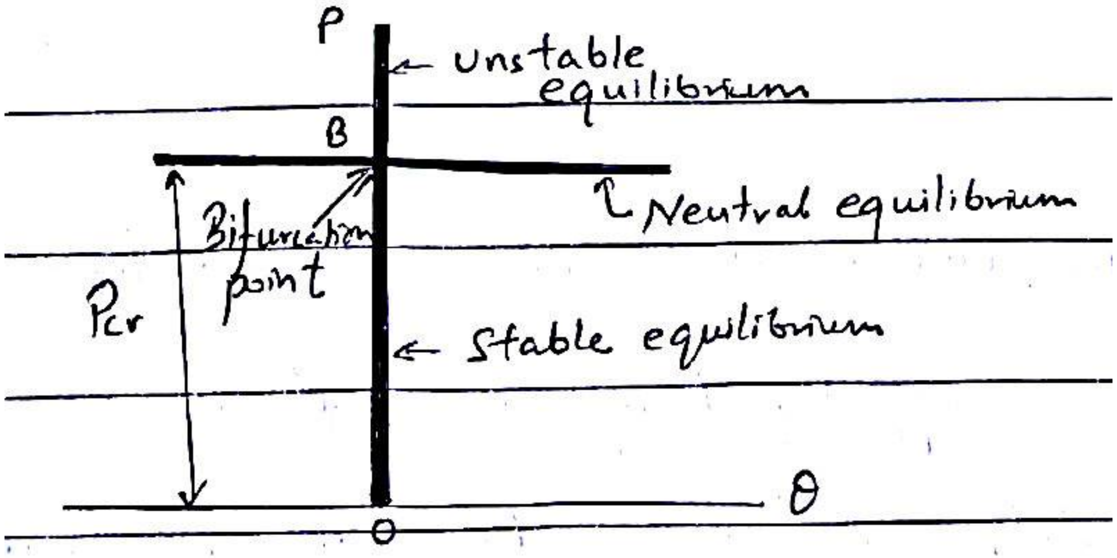
(3) If restoring moment is less than distorting moment then column is in unstable equilibrium.

$$KL^2\theta < PL\,\theta$$

$$KL < P \quad \text{(condition for unstable equilibrium)}$$

These equilibrium relationships are shown in the graph of P versus θ below.

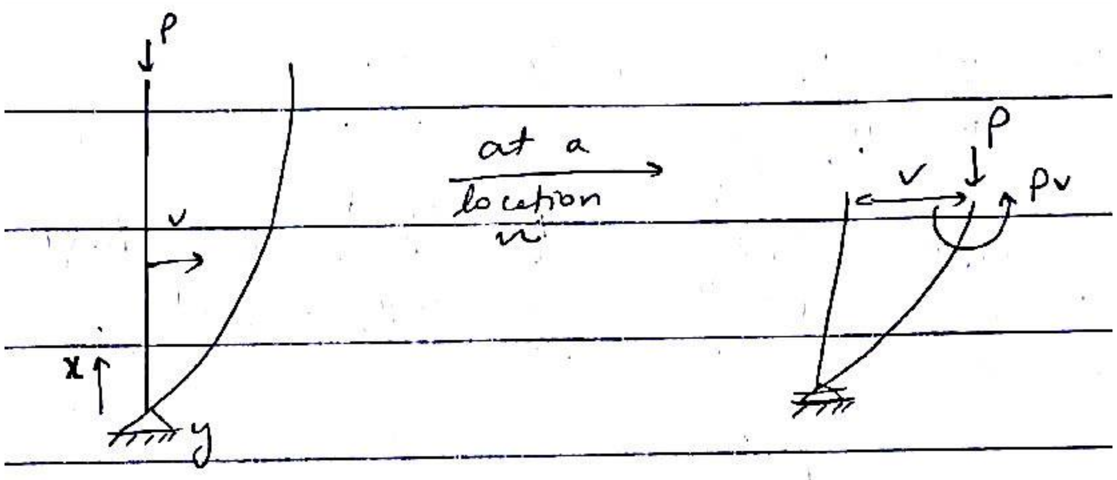
IITians GATE CLASSES
BANGALORE
Visit us: www.iitiansgateclasses.com
Mail us: info@iitiansgateclasses.com



- **Euler buckling theory**

This theory is applicable to materials following Hook's law.

Consider the loaded column as shown below,



$$M = - EI\,\frac{d^2V}{dx^2}$$

$$PV = - EI \frac{d^2V}{dx^2}$$

$$\frac{d^2V}{dx^2} + \frac{P}{EI}V = 0$$

<u>Solution: -</u>

$$V = A\sin kx + B\cos kx$$

$$\text{Where } k = \sqrt{\frac{P}{EI}}$$

To find A and B, we use boundary condition.

- **Hinged – Hinged**

V(0) = 0; V(l) = 0

∴ B = 0

A$\sin kl$ = 0

∴ $\sin kl$ = nπ where n = 1, 2, 3.......

k = $\frac{n\pi}{l}$

$$\mathbf{P_{cr} = \frac{n^2\pi^2 EI}{l^2}}$$

Here I should be the minimum principal inertia.

Also,

I = Ar^2 → where r is radius of gyration.

$P_{cr} = \frac{\pi^2 EA}{\left(\frac{l}{r}\right)^2}$ → where $\frac{l}{r}$ is slenderness ratio.

And,

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$$

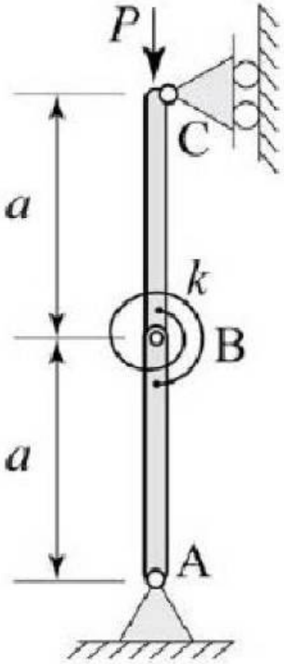
- **Similarly for other boundary conditions**

Boundary Condition	L_{eq}	P_{cr}
• **Hinged-Hinged**	L	$\frac{\pi^2 EI}{L^2}$
• **Fixed-Fixed**	L/2	$\frac{4\pi^2 EI}{L^2}$
• **Fixed-Free**	2L	$\frac{\pi^2 EI}{4L^2}$
• **Fixed-Hinged**	L/√2	$\frac{2\pi^2 EI}{L^2}$

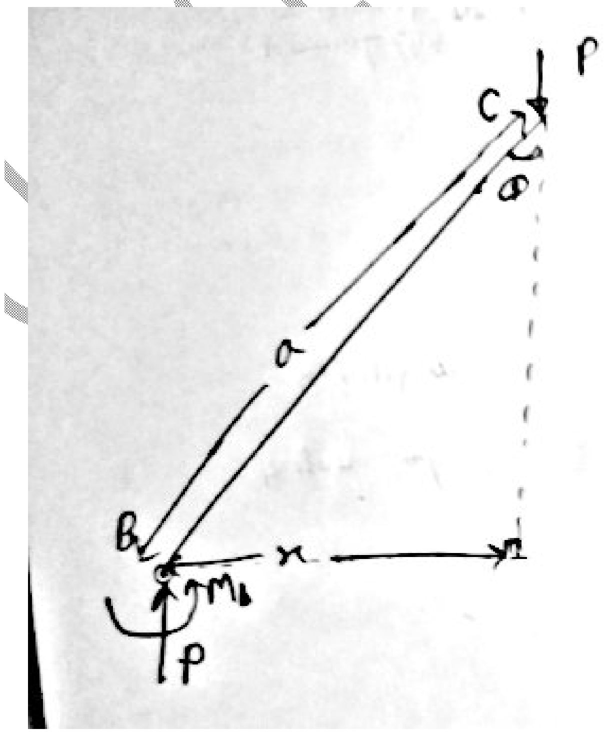
Problems

1.

Two massless rigid bars, each of length $a = 0.5\text{m}$, are connected by a rotational spring having stiffness $k = 1000$ N.m/rad. Find the buckling load P (in kN).



Sol. 4 kN



x = aθ

k = 1000 N.m/rad

a = 0.5 m

D.M. = P.aθ

R.M. = k.2θ

$\sum M_b$ = 0

R.M. - D.M. = 0

k.2θ – P.aθ = 0

So $P_{cr} = \frac{2k}{a} = \frac{2\times1000}{0.5}$ = 4 kN

2.

For a long slender column of uniform cross section, the ratio of critical buckling load for the case with both ends clamped to the case with both ends hinged is

(A) 1 (B) 2 (C) 4 (D) 8

Sol. C

$P_{cr1} = \frac{4\pi^2 EI}{l^2}$ (for both ends fixed)

$P_{cr2} = \frac{\pi^2 EI}{l^2}$ (for both ends pined)

So $\frac{P_{cr1}}{P_{cr2}} = 4$

3.

The buckling load for a simply supported column of rectangular cross section of dimensions 1 cm×1.5 cm and length 0.5 m made of steel ($E = 210\times10^9$ N/m^2) is approximately

(A) 10 kN (B) 4 kN (C) 23 kN (D) 46 kN

Sol. A

For a simply supported beam,

$P_{cr} = \frac{\pi^2 EI}{l^2}$

E = 210× 10^9 N/m^2

l = 0.5 m

$I = \frac{bd^3}{12} = \frac{(1.5\times10^{-2})\times(1\times10^{-2})^3}{12} = 0.125\times 10^{-8}\ m^4$

Then, P_{cr} = 10.36 kN

4.

A column of length l and flexural rigidity EI, has one end fixed and the other end hinged. The critical buckling load for the column is

(A) $\frac{\pi^2 EI}{(0.5l)^2}$ (B) $\frac{\pi^2 EI}{(0.7l)^2}$ (C) $\frac{\pi^2 EI}{l^2}$ (D) $\frac{\pi^2 EI}{(2l)^2}$

Sol. B

For one end fixed and other is hinged,

L_{eq} = L/$\sqrt{2}$

$P_{cr} = \frac{\pi^2 EI}{(0.7l)^2}$

5.

Consider a steel (Young's modulus $E = 200$ GPa) column hinged on both sides. Its height is 1.0 m and cross-section is 10 mm × 20 mm. The lowest Euler critical buckling load (in N) is ___________

Correct Answer :
3285 to 3295

Sol. 3289.73 N

$P_{cr} = \frac{\pi^2 EI}{l^2}$ (for both side hinged)

E = 200× 10^3 N/mm^2 , l = 1m = 1000 mm, $I = \frac{bd^3}{12} = \frac{20 \times 10^3}{12} = 1666.66\ mm^4$

So, P_{cr} = 3289.73 N

6.

A simply supported slender column of square cross section (width=depth=d) has to be designed such that it buckles at the same instant as it yields. Length of the column is given to be 1.57 m and it is made of a material whose Young's modulus is 200 GPa and yield stress is 240 MPa. The width, d, of the column (in cm) should be ________

Sol. 5.99 cm

$P_{cr} = \frac{\pi^2 EI}{l^2}$ (for simply supported column)

l = 1.57 m, E = 200GPa, σ_{yield} = 240 MPa

$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{l^2 . d^2}$, $I = \frac{BD^3}{12} = \frac{d^4}{12}$

$\sigma_{cr} = \frac{\pi^2 E}{l^2 . d^2} \times \frac{d^4}{12} = \sigma_{yield}$

d = $\sqrt{\left(\frac{\sigma_{yield} \times 12 \times l^2}{\pi^2 E}\right)}$ = 0.0599 m = 5.99 cm

7.

For a slender steel column of circular cross-section the critical buckling load is P_{cr}. If the diameter of the column is doubled (keeping other material and geometrical parameters same), then the critical buckling load of the column is

(A) $P_{cr}/16$ (B) $8P_{cr}$ (C) $2P_{cr}$ (D) $16P_{cr}$

Sol. D

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$I = \frac{\pi d^4}{64}$$

$$P_{cr} = \frac{\pi^2 E}{l^2} \times \frac{\pi d^4}{64}$$

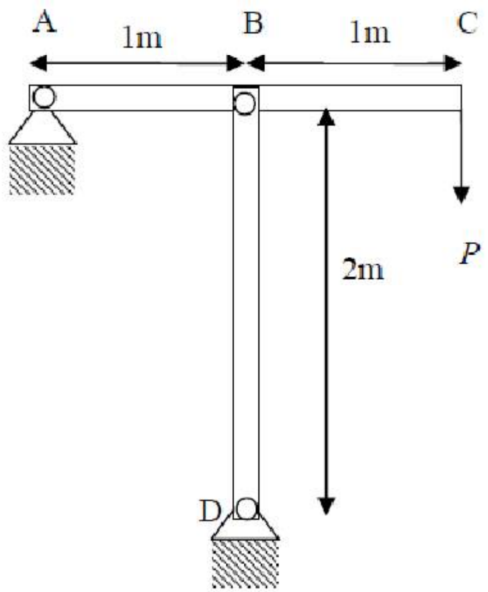
$$P_{cr} \propto d^4$$

$\frac{P_{cr,2}}{P_{cr,1}} = \left(\frac{d_2}{d_1}\right)^4 = 2^4 = 16$

$P_{cr,2} = 16P_{cr,1}$

8.

In the steel structure (Young's modulus = 200 GPa) shown in the figure, all members have a circular cross-section of radius 10 mm. Column BD is pinned at B and D. The support at A is hinged. The minimum value of load *P* at which the column BD may buckle in *Newtons* is approximately ____________



Sol. 1937.8 N

Lets R is load applied on column BD.

E = 200 GPa, radius of member, r = 10 mm, $I = \frac{\pi d^4}{64}$

$\sum M_A = 0 = R\times 1 - P\times 2$

$R = 2P = \frac{\pi^2 EI}{l^2} =$

$P = \frac{1}{2} \times \frac{\pi^2 E}{l^2} \times \frac{\pi d^4}{64} = 1937.8$ N

9.

A structural member of rectangular cross-section $10mm \times 6mm$ and length $1m$ is made of steel (Young's modulus is $200GPa$ and coefficient of thermal expansion is $12 \times 10^{-6} / {}^{\circ}C$). It is rigidly fixed at both the ends and then subjected to a gradual increase in temperature. Ignoring the three dimensional effects, the structural member will buckle if the temperature is increased by $\Delta T\ {}^{\circ}C$ which is

(A) 19.74 (B) 9.87 (C) 78.96 (D) 39.48

Sol. B

l = 1m, E = 200GPa, α = 12×10^{-6} /°C

$I = \frac{b \times d^3}{12} = \frac{10 \times 6^3}{12} = 180\ mm^4$

$P_{cr} = \frac{4\pi^2 EI}{l^2}$ (for both side fixed beam)

$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{4\pi^2 EI}{l^2 . A} = \alpha \Delta T E = \sigma_T$

$\Delta T = \frac{4\pi^2 I}{l^2 . A . \alpha} = 9.89\ ^\circ C$

10.

A 200 mm long simply-supported column has a 5 mm $\times$ 10 mm rectangular cross section. The Young's modulus of the material, $E = 200\ GPa$. Assuming a factor of safety of 2.5 corresponding to the buckling load, the maximum load (in N) the column can support in compression is ________.

Sol. 2056.03 N

l = 200 mm, E = 200 GPa, fos = 2.5

fos = $\frac{yield\ load}{applied\ load}$

$yield\ load,\ P_{cr} = \frac{\pi^2 EI}{l^2}$

Then, applied load, P = $\frac{\pi^2 EI}{l^2.fos}$

$I = \frac{b \times d^3}{12} = \frac{10 \times 5^3}{12}$ = 104.16 mm^4

P = 2056.03 N

11.

A solid bar of uniform square cross-section of side b and length L is rigidly fixed to the supports at the two ends. When the temperature in the rod is increased uniformly by Tc, the bar undergoes elastic buckling. Assume Young's modulus E and coefficient of thermal expansion α to be independent of temperature. The coefficient of thermal expansion α is given by

(A) $\frac{3\pi^2 b^2}{T_c L^2}$ (B) $\frac{\pi^2 b^2}{T_c L^2}$ (C) $\frac{\pi^2 b^2}{2T_c L^2}$ (D) $\frac{\pi^2 b^2}{3T_c L^2}$

Sol. D

$P_{cr} = \frac{4\pi^2 EI}{l^2}$ (for both side fixed bar)

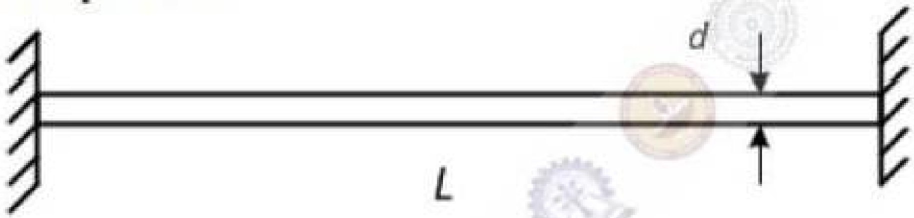
$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{4\pi^2 EI}{l^2.b^2} = \alpha \Delta T E$

$I = \frac{b \times d^3}{12} = \frac{b^4}{12}$

So, $\alpha = \frac{4\pi^2}{l^2.b^2.T_c} \times \frac{b^4}{12} = \frac{1}{3}.\frac{\pi^2.b^2}{l^2.T_c}$

12.

An initially stress-free massless elastic beam of length L and circular cross-section with diameter d ($d \ll L$) is held fixed between two walls as shown. The beam material has Young's modulus E and coefficient of thermal expansion α.



If the beam is slowly and uniformly heated, the temperature rise required to cause the beam to buckle is proportional to

(A) d (B) d^2 (C) d^3 (D) d^4

Sol. B

Area, A = $\frac{\pi d^2}{4}$

$I = \frac{\pi d^4}{64}$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{4\pi^2 E.\frac{\pi d^4}{64}}{l^2.\frac{\pi d^2}{4}} = \alpha \Delta T E$$

$$So,\ \Delta T \quad \propto \ d^2$$