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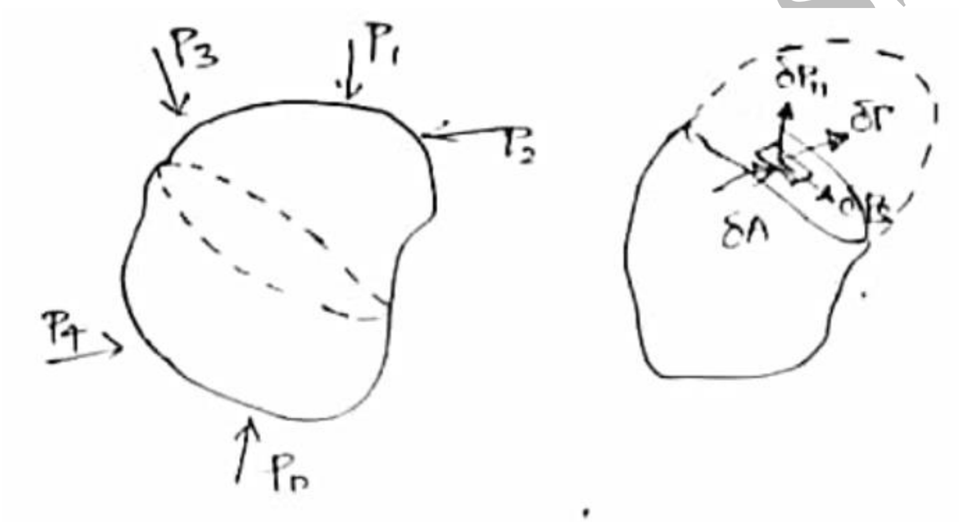
Basic elasticity

Stress:-

When a body undergoes deformation under the application of external force, a restoring/resistance force is induce within the body. The intensity of force,(i.e), restoring force per unit area is termed as stress.

$$stress\ (\sigma) = \lim_{\delta A \to 0} \frac{\delta P}{\delta A}$$



Under complex loading, for any given small area, δA the resultant force can be at any inclination. This the resultant force (here stress) is resolved in two components.

(1) Normal Stress:- (Normal to plane)

$$\sigma = \lim_{\delta A \to 0} \frac{\delta P}{\delta A}$$

(2) Shear Stress:- (Parallel to plane)

$$\tau = \lim_{\delta A \to \infty} \frac{\delta Ps}{\delta A}$$

Stress is a tensor quantity (2^{nd} order tensor) (i.e), it depends on magnitude, direction and plane in which it acts.

Normal stress (σ) can be tensile or compressive is nature depends on loading.

Generally, normal stress pointing away from plane is considered as tensile stress (+ve) while normal stress pointing towards the plane is considered as compressive stress(-ve).

Normal stress is also termed as direct stress.

Normal and Direction of Stresses:-

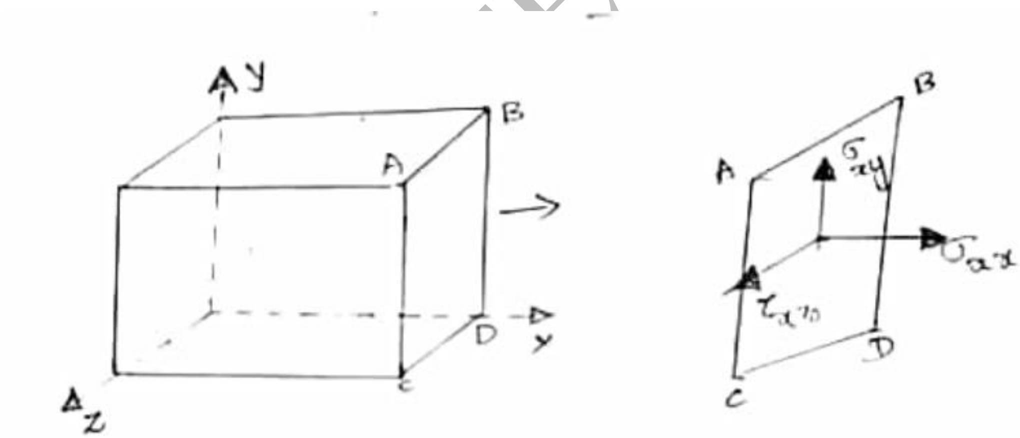
In tonsorial notation the stress is generally termed as σ_{ij} or τ_{ij} two suffix for 2nd order tensor

When

i –indicates plane in which it acts

j – indicates direction of action

for eg



In above figure plane ABCD is ‘ X-plane’. There is a normal force and two shear stress in a plane of ‘ 3-D structure’. The normal stress in plane ABCD is noted as σ_{xx}. Similarly, shear stress τ_{xy} in y-direction and τ_{xz} in z-direction.

Some time, σ_{xx} is also termed with σ_x (single σ_x).

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Direction:-

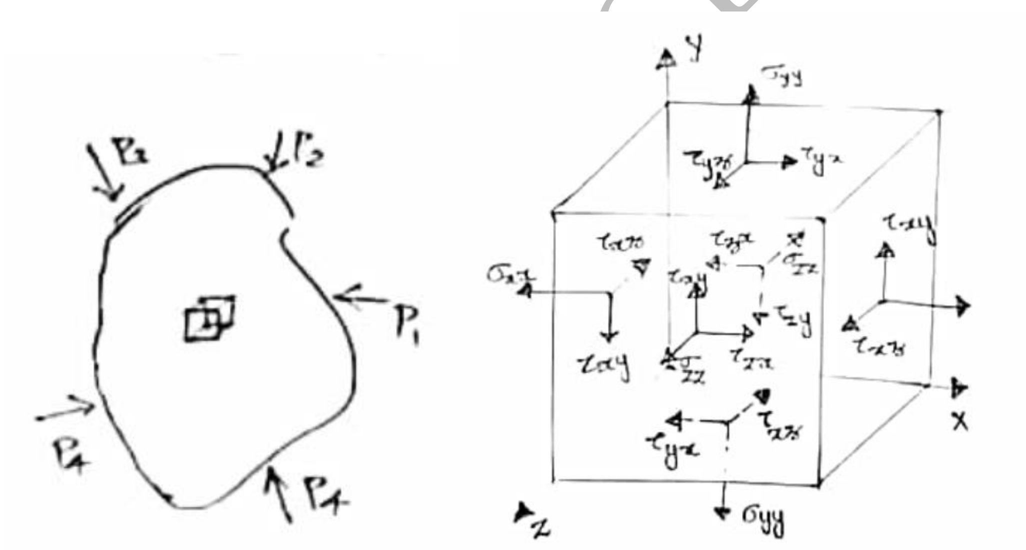
The normal stresses are defined as positive when they are directed away from their related surface.

The direction of shear stress depends on the co-ordinate system considered.

If the tensile stress is in positive direction of the axis (say x), the shear stress are positive in other two positive direction of axis (y and z), while if tensile stress in opposite direction of axis (-x), the positive shear stress are in direction opposite to positive direction of axis (-y and –z).

Stresses in 3-D Body:-

If any 3D body is subjected to external load it undergoes deformations which include strains and stresses. Consider a small (infinitesimal) particle of a 3-D body.



There are 3-planes in a cubical partial (X,Y,Z). in each plane (one normal stress and two shear stress).

There are total 9-stresses in a 3-D body. In terms of matrix,

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$$[\sigma]=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}$$

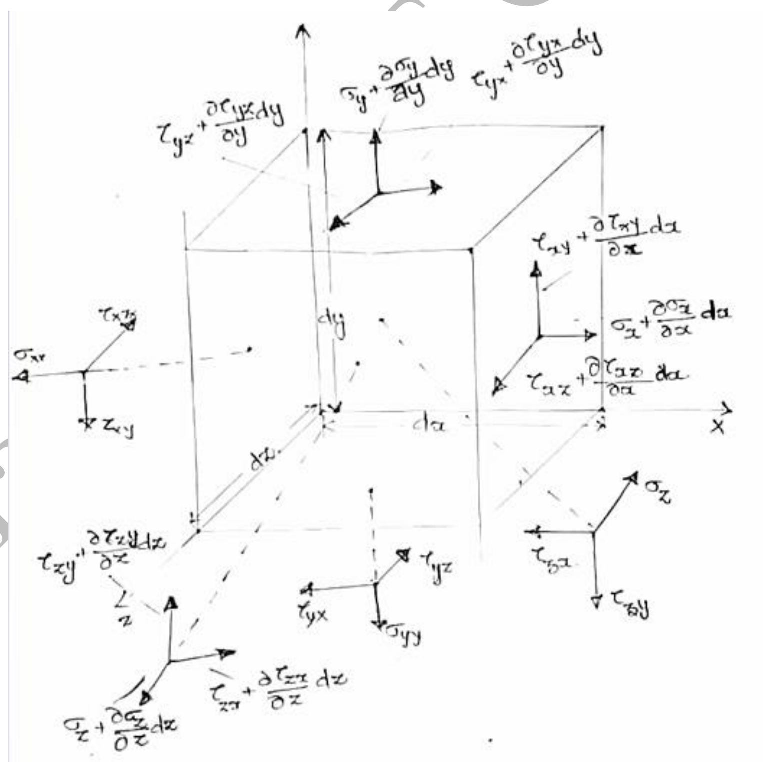
Row indicate plane

Column indicate direction

Equation of Equilibrium:-

Consider again the infinitesimal element of a 3-D body subjected to external loading. Assume the body to be in equilibrium under the external loading, here the small element is also in equilibrium.

Considering the variation of stress along the element as well. Let the body force in x,y and z direction in X,Y and Z.



Taking moment about an axis through centre of element and parallel to z-axis.

$$(\tau_{xy} dydz)\frac{dx}{g} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dydz\frac{dx}{g} - \tau_{yx} dx\, dz\frac{dy}{g} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dxdz\frac{dy}{g} = 0$$

Ignoring higher order term and dividing by dx dy dz;

$$\tau_{xy} = \tau_{yx}$$

Similarly,

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

Complementary nature of shear stress.

A shear stress acting on a given plane $(\tau_{xy}, \tau_{xz}, \tau_{zy})$ is always accompanied by an equal complementary shear stress $(\tau_{yx}, \tau_{zx}, \tau_{zy})$ acting on a plane perpendicular to the given plane and is opposite sense.

Now considering force equilibrium in x-direction,

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dydz - \sigma_x dydz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dxdz - \tau_{yx} dxdz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dxdy - \tau_{zx} dxdy + X\, dxdydz = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

Using complementary shear stress,(i.e), $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + Y = 0$$

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$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + Z = 0$$

The above equ of equailibrium must be satisfied at all interior points in a deformable body a 3-D force system.

Strain:-

Strain is defined as change in length to original length.

Normal strain due to normal stress.

Shear strain due to shear stress.

Normal Strain:-

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z}$$

Shear strain:-

$$\gamma_{zx} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

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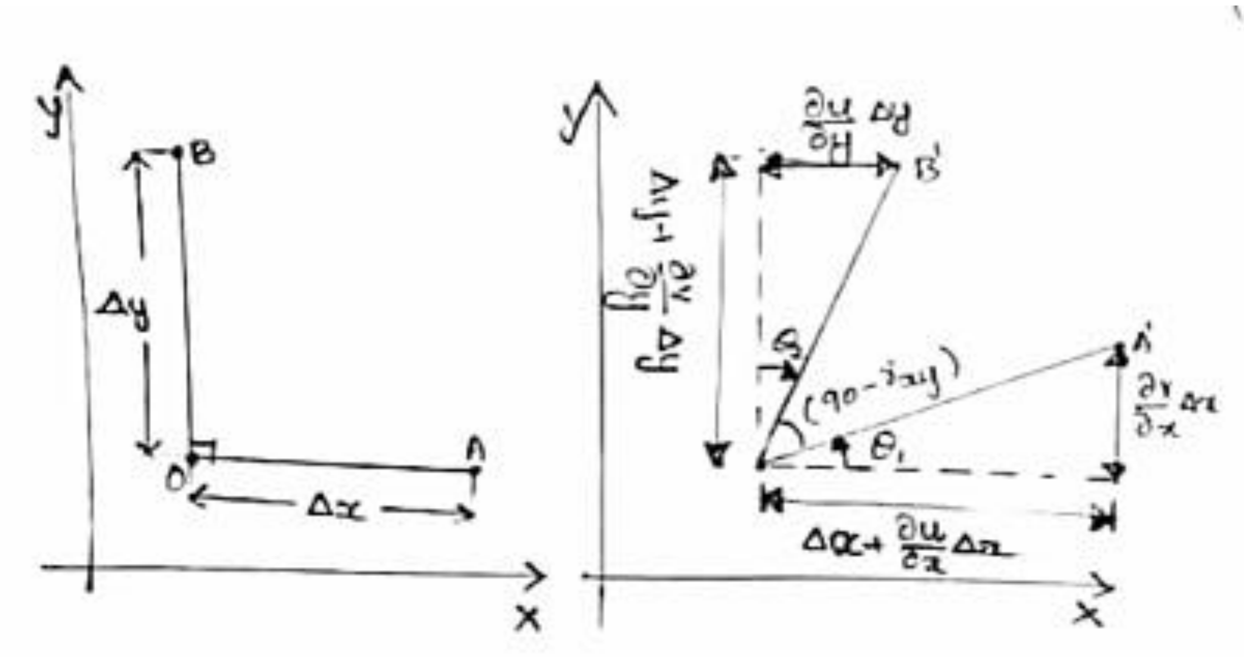

$$\varepsilon_x = \frac{O'A' - OA}{OA} = \frac{\partial u}{\partial x}$$

Similarly,

$$\varepsilon_y = \frac{O'B' - OB}{OB} = \frac{\partial v}{\partial y}$$

Also,

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Strain transformation:-

$$\begin{Bmatrix} \varepsilon_s \\ \varepsilon_t \\ \frac{\gamma_{st}}{2} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \frac{\gamma_{xy}}{2} \end{Bmatrix}$$

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Principal strain:-

$$\varepsilon_{I,II} = \frac{\varepsilon_x - \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{(\varepsilon_{xx} - \varepsilon_{yy})}\right)$$

Max shear strain:-

$$\frac{\gamma_{max}}{g} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\theta_3 = \frac{1}{2}\tan^{-1}\left(-\frac{(\varepsilon_{xx} - \varepsilon_{yy})}{\gamma_{xy}}\right)$$

Stress strain Relationship:-

Assumption:-

Material is isotropic and homogeneous

The structure is deformed within elastic limit

Hook's Law:-

When a 1-D body (say bar) is loaded axially, it undergoes deformation (ΔL), hence, strain and stresses induced Hook's law states whenever a body is subject to external loading within elastic limit, the stresses induced is directly proportional strain,(i.e), ($\sigma \propto \varepsilon$)

$$\sigma = E\varepsilon$$

Where,

E- Young's modules of elasticity.

$$\text{Poisson's Ratio } (\gamma) = \frac{-Lateral\ strain}{Longitudinal\ strain} = -\frac{\varepsilon_y}{\varepsilon_x}$$

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$$\gamma = -\frac{\varepsilon_y}{\varepsilon_x}$$

$$-1 < \gamma < 0.5$$

Suppose a 3-D body is subjected to uniaxial loading in X-direction.

$$E = \frac{\sigma_x}{\varepsilon_x} \Rightarrow \varepsilon_x = \frac{\sigma_x}{E} \rightarrow (1)$$

Similarly,

$$\varepsilon_z = -\frac{\gamma\sigma_x}{E} \rightarrow (2)$$

$$\varepsilon_y = -\frac{\gamma\sigma_y}{E} \rightarrow (3)$$

Similarly, for uniaxial loading in Y-direction:-

$$\varepsilon_y = \frac{\sigma_y}{E} \quad ; \quad \varepsilon_x = -\frac{\gamma\sigma_y}{E} \quad ; \quad \varepsilon_z = -\frac{\gamma\sigma_y}{E} \rightarrow (4)$$

In Z- direction,

$$\varepsilon_z = \frac{\sigma_z}{E} \quad ; \quad \varepsilon_x = -\frac{\gamma\sigma_x}{E} \quad ; \quad \varepsilon_y = -\frac{\gamma\sigma_y}{E} \rightarrow (5)$$

For multidirectional loading, combining all equations:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \gamma(\sigma_y - \sigma_z)\right] \rightarrow (A)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \gamma(\sigma_x - \sigma_z)\right] \rightarrow (B)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \gamma(\sigma_x - \sigma_y)\right] \rightarrow (C)$$

Similar to Hook's law for normal stress, Hook's law in shear component is, $\tau \propto \gamma$; $\tau = G\gamma$ Where, G – shear modulus