ASTRODYNAMICS / ORBITAL MECHANICS

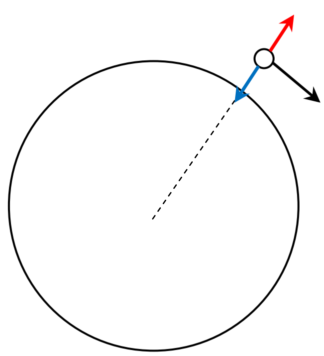
Study of the motion of man-made objects in space subject to both natural and artificially induced forces (different from the parent science celestial mechanics, where there is no human intervention).

An astrodynamicist's job is to determine the trajectory to accomplish the desired mission goals within the constraints of limits imposed by physics and launch vehicle performance.

A review of what we know from high school physics

The orbital velocity:

Gravitational force = GmM/R^2 = mg Centrifugal force = mV^2/R



$mg = mV^2/R$

$V_{orbital} = \sqrt{(gR)}$

$= 8$ *km/s*

Escape velocity:

$V_{orbital} = \sqrt{(2gR)} = 11.2 km/s$

Equation of Motion:

The equation of motion for single body about a central gravitational field is given by

$$\vec{F} = m\ddot{\vec{r}} = -\frac{GMm}{|\vec{r}|^3}\vec{r}$$

or $$\ddot{\vec{r}} + \frac{GM}{|\vec{r}|^3}\vec{r} = \ddot{\vec{r}} + \frac{\mu}{|\vec{r}|^3}\vec{r} = 0 \text{ where } \mu = GM$$

Kepler's law of planetary motion (~1600AD, Germany based on data of Tacho Brahe)

1. Planets move around Sun in elliptical orbits with Sun as one of the foci.

2. The line joining the planet and the Sun sweeps equal area in equal times.

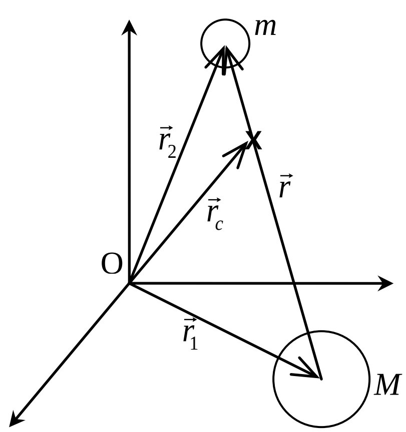
3. The square of the time period of revolution of a planet about the Sun is proportional to the major axis of the ellipse

Fundamentals of Orbital Mechanics

Two-Body motion

The basis of astrodynamics is Newton's universal law of gravitation which describes the force between two point masses *M* and *m* separated by a distance $|\vec{r}|$ along the vector $\vec{r}$.

$$F = -\frac{GMm}{|\vec{r}|^3}\vec{r}$$



The equations of motion for the system can be written as:

$$\vec{F}_M = M\ddot{\vec{r}}_1 = -\frac{GMm}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \qquad \text{------(1)}$$

$$\vec{F}_m = m\ddot{\vec{r}}_2 = -\frac{GMm}{\left|\vec{r}_1 - \vec{r}_2\right|^3}\left(\vec{r}_2 - \vec{r}_1\right) \qquad \text{------(2)}$$

The information available forbids determination of all the constants of integration (12 in number) that are obtained upon integration of equations ‘1’ and ‘2’. However, the relative motion between the two bodies is amenable to solution.

Adding Equn. (1) and Equn. (2)

$$M\ddot{\vec{r}}_1 + m\ddot{\vec{r}}_2 = 0 \qquad \text{-----(3a)}$$

Let the position vector to the center of mass of the *m* and *M* mass system be $\vec{r}_c$

$$\vec{r}_c = \frac{M\vec{r}_1 + m\vec{r}_2}{M + m}$$

Equn. (3) becomes $\boxed{\ddot{\vec{r}}_c = 0}$ -----(3b)

i.e. the centre of mass of the two body system is un-accelerated and thus can serve as the origin of an inertial frame.

Now subtracting $m \times$ Equn. (1) form $M \times$ Equn.(2)

$$Mm\left(\ddot{\vec{r}}_2 - \ddot{\vec{r}}_1\right) = -\frac{GMm\left(M + m\right)}{\left|\vec{r}_1 - \vec{r}_2\right|^3}\left(\vec{r}_2 - \vec{r}_1\right)$$

$$\Rightarrow \ddot{\vec{r}} = -\frac{G\left(M + m\right)}{\left|\vec{r}\right|^3}\vec{r} = -\frac{\mu}{\left|\vec{r}\right|^3}\vec{r}, \quad where\ \mu = G(M + m)$$ is called the reduced mass

$$\ddot{\vec{r}} + \frac{\mu}{\left|\vec{r}\right|^3}\vec{r} = 0 \qquad \text{------(4)}$$

This equation is same as the equation of motion for a single mass about a central gravitational field, except that the present equation, equn. (4) describes the motion of body ‘m’ relative to body M. Further for M>>m the center of mass of the system shifts almost to the center of mass ‘M’ and thus reduce the two boy problem to that of motion of single body about central gravitational field.

Linear momentum

Linear momentum of mass ‘m’ and ‘M’ are $\vec{p}_M = M\dot{\vec{r}}_1$ and $\vec{p}_m = m\dot{\vec{r}}_2$ respectively

Linear momentum of the system $\vec{p} = \vec{p}_M + \vec{p}_m = M\dot{\vec{r}}_1 + m\dot{\vec{r}}_2 = const.$ as $\dot{\vec{r}}_c = 0$

Angular momentum

The angular momentum of the system is $\vec{H} = \vec{H}_M + \vec{H}_m$ where $\vec{H}_M = \vec{r}_1 \times M\dot{\vec{r}}_1$ and $\vec{H}_m = \vec{r}_2 \times m\dot{\vec{r}}_2$

Now total torque on the system will result in the rate of change of the momentum of the system.

i.e. $\tau = \sum_i \tau_i = \sum_i \vec{r}_i \times F_i = \sum_i \frac{d}{dt}\left(\vec{r}_i \times m_i\dot{\vec{r}}_i\right) = \sum_i \dot{\vec{H}}_i = \dot{\vec{H}}$

$$\vec{\tau} = \sum_i \vec{\tau}_i = \sum_i \vec{r}_i \times \vec{F}_i = \vec{r}_1 \times \frac{GMm}{|\vec{r}|^3}\left(\vec{r}_1 - \vec{r}_2\right) + \vec{r}_2 \times \frac{GMm}{|\vec{r}|^3}\left(\vec{r}_2 - \vec{r}_1\right) = -\vec{r}_1 \times \vec{r}_2 - \vec{r}_2 \times \vec{r}_1 = 0$$

One can also show that $\dot{\vec{H}} = \sum_i \dot{\vec{H}}_i = \sum_i \frac{d}{dt}\left(\vec{r}_i \times m_i\dot{\vec{r}}_i\right)$

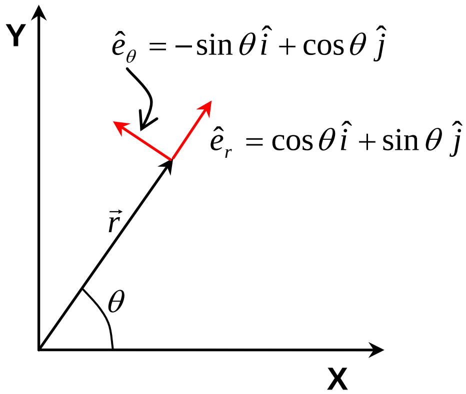
$$\dot{\vec{H}}_M = \dot{\vec{r}}_1 \times M\dot{\vec{r}}_1 + \vec{r}_1 \times M\ddot{\vec{r}}_1 = 0 + \vec{r}_1 \times \frac{GMm}{|\vec{r}|^3}\vec{r} \;\; \& \;\; \dot{\vec{H}}_m = \dot{\vec{r}}_2 \times m\dot{\vec{r}}_2 + \vec{r}_2 \times m\ddot{\vec{r}}_2 = 0 - \vec{r}_2 \times \frac{GMm}{|\vec{r}|^3}\vec{r}$$

$$\dot{\vec{H}} = \left(\vec{r}_1 - \vec{r}_2\right) \times M\,\frac{\mu}{|\vec{r}|^3}\vec{r} = \vec{r} \times \frac{GMm}{|\vec{r}|^3}\vec{r} = 0$$

$\vec{H} = \text{constant vector}$ and perpendicular to the plane containing $\vec{r}$ and $\dot{\vec{r}}$, therefore, the motion described by the bodies will be two-dimensional along the plane orthogonal to the angular velocity vector.

The equation of motion of mass ‘m’ about ‘M’ described by the ODE

$\boxed{\ddot{\vec{r}} + \frac{\mu}{|\vec{r}|^3}\vec{r} = 0,}$ $\quad where\ \mu = G(M+m) \quad and \quad \vec{r} = \vec{r}_2 - \vec{r}_1$ can be expressed in polar co-ordinates as show below



$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta$$
$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$$

$$\vec{r} = r\,\hat{e}_r$$
$$\dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$
$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta$$

Thus equn. (4), i.e. equation of motion in polar co-ordinates becomes

$$\left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta + \frac{\mu}{r^2}\hat{e}_r = 0$$

$$\left(\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2}\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta = 0$$

$$radial\ component: \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0 \qquad and \qquad circumferential\ \ component: 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

Let us look at the circumferential component

The circumferential component can be written as: $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

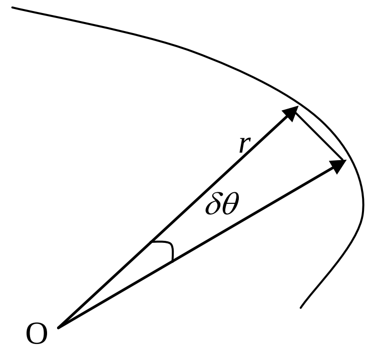
or $r^2\dot{\theta} = const. = h$

where h is angular momentum/mass, or $h = \vec{r} \times \dot{\vec{r}} = \left(r\hat{e}_r\right) \times \left(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta\right) = r^2\dot{\theta}\,\hat{e}_z$

$$\frac{\delta A}{\delta t} = \frac{1}{2} r \times r \frac{\delta\theta}{\delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = const.$$

This is Kepler's second law



Now we look at the radial component

$$radial\ component: \ \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$$

$$\dot{r}\left(\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2}\right) = 0 \qquad OR \qquad \dot{r}\ddot{r} - \dot{r}\frac{h^2}{r^3} + \dot{r}\frac{\mu}{r^2} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} - \frac{\mu}{r}\right) = 0 \quad OR \quad \frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} - \frac{\mu}{r} = const.$$

$$OR \quad \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{\mu}{r} = const.$$

$$\underbrace{\left(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right)}_{kinetic\ energy\,/\,mass} + \underbrace{\left(-\frac{\mu}{r}\right)}_{potential\ energy\,/\,mass} = const. = \underbrace{E_T}_{total\ energy\,/\,mass} \quad (Orbital\ Energy)$$

Total energy of the two body system remains a constant.

Further Reading:

1. Mechanics & Thermodynamics of Propulsion (2nd Ed.) by Hill & Peterson. Chapter#10 section10.6

2. Space Flight Dynamics (2nd Edition) by William E Wiesel, Tata McGraw Hill, Chapter # 1 and 2.

3. Introduction to Space Flight by Francis J Hale, Prentice Hall, Chapter # 2

4. Orbital Mechanics by V A Chobotov, Chapter # 3