Equation of Trajectory

From radial component we get $\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad \ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

$$-\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{h^2}{r^3} + \frac{\mu}{r^2} = 0$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} - \frac{\mu}{h^2} = 0 \quad let\ \frac{1}{r} - \frac{\mu}{h^2} = Z \qquad \Rightarrow \frac{d^2 Z}{d\theta^2} + Z = 0$$

$$Z = A\cos\theta + B\sin\theta = C\cos(\theta - \psi), \quad where\ C = \sqrt{A^2 + B^2}\ and\ \psi = \tan^{-1}\left(\frac{B}{A}\right)$$

$$\frac{1}{r} = \frac{\mu}{h^2} + C\cos(\theta - \psi) = \frac{\mu}{h^2}\left(1 + e\cos(\theta - \psi)\right) \quad where\ e = \frac{Ch^2}{\mu}$$

The equation of trajectory of mass ‘m’ about ‘M’ is given by: $r = \frac{h^2/\mu}{\left(1 + e\cos(\theta - \psi)\right)}$

The equation above represents a class of curves referred to as conic curves. We take a look at the mathematical description of the conic curves next.

The conic section

The family curves obtained when a plane cuts a conical surface is called conics. The different conics are shown below.

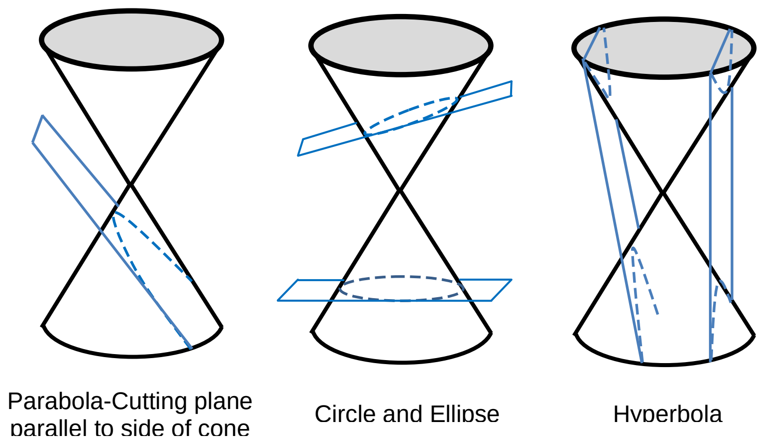

Figure 1

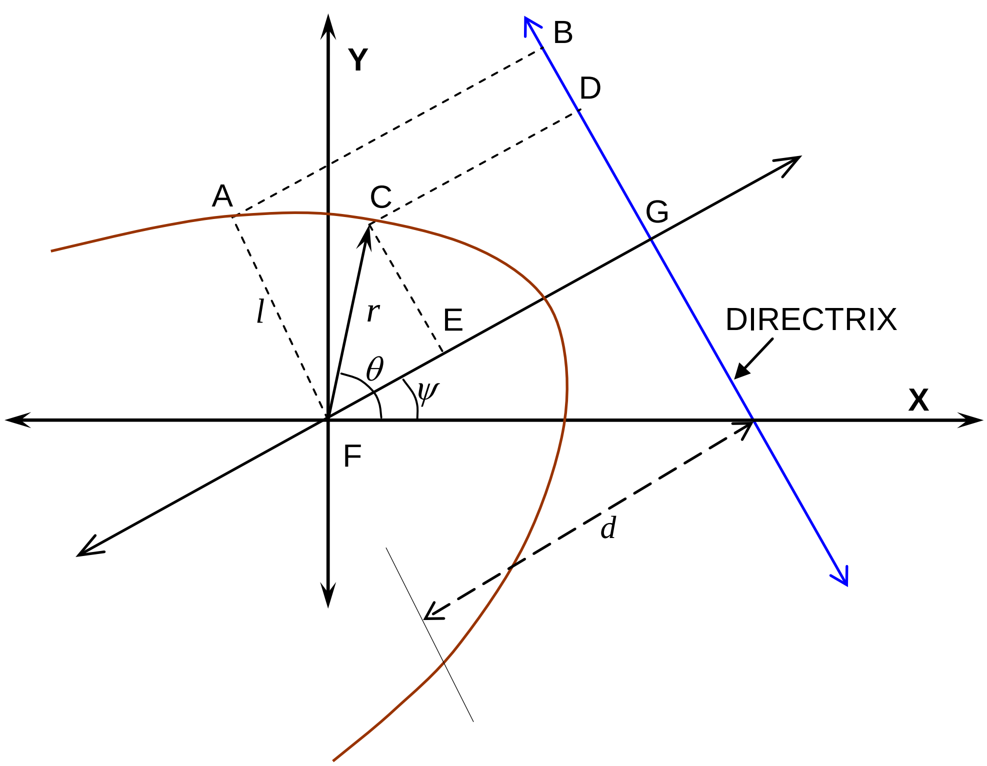

Figure 2

Mathematically defining, a conic is a locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is a positive constant. The ratio is the eccentricity of the conic, the fixed point the focus and the fixed line the directrix (Fig. 2).

$$r = CF = e\,CD = e\left(FG - FE\right) = e\left(AB - FE\right) = l - e\,r\cos\left(\theta - \psi\right)$$

$$r = \frac{l}{1 + e\cos(\theta - \psi)} = \frac{ed}{1 + e\cos(\theta - \psi)}$$

In the cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section, and all conic sections arise in this way. If the equation is of the form

$$ax^2 + 2cxy + by^2 + 2dx + 2ey + c = 0$$

then:

if $c^2 = ab$, the equation represents a parabola

if $c^2 < ab$ and $a \neq b$ and/or $c \neq 0$, the equation represents an ellipse

if $c^2 > ab$, the equation represents a hyperbola

if $c^2 < ab$ and $a = b$ and $c = 0$, the equation represents a circle

Trajectory Classification

Equation of trajectory for a two body problem $r = \dfrac{h^2/\mu}{(1 + e\cos(\theta - \psi))}$

Let $\psi = 0$, then $r = \dfrac{h^2/\mu}{1 + e\cos\theta}$ (see figure below)

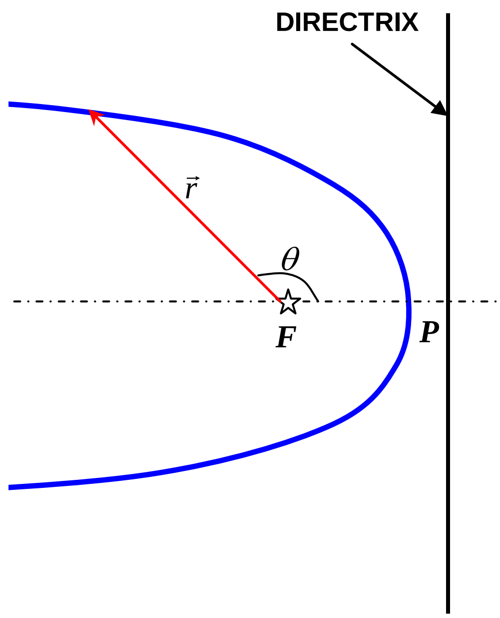

Figure 3

$$\underbrace{\left(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right)}_{kinetic\ energy/mass} + \underbrace{\left(-\frac{\mu}{r}\right)}_{potential\ energy/mass} = const. = \underbrace{E_T}_{total\ energy/mass} \quad (Orbital\ Energy)$$

or $E_T = \frac{1}{2}V^2 - \frac{\mu}{r} = const.$, where $V^2 = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2$

Now the constant value of E_T can be evaluated at $\theta = 0,$ where $r = r_{\min}$ and $\dot{r} = 0$

$E_T = \frac{1}{2}r_{\min}{}^2\dot{\theta}^2 - \frac{\mu}{r_{\min}}$, also $h = r^2\dot{\theta} = const. = r_{\min}{}^2\dot{\theta}$ and $r_{\min} = \dfrac{h^2/\mu}{1+e}$

$$\therefore \quad E_T = \frac{1}{2} r_{\min}{}^2 \dot{\theta}^2 - \frac{\mu}{r_{\min}} = \frac{1}{2}\frac{h^2}{r_{\min}^2} - \frac{\mu}{r_{\min}} = \frac{\mu(e-1)}{2r_{\min}} = -\frac{\mu^2(1-e^2)}{2h^2}$$

$$E_T = -\frac{\mu^2(1-e^2)}{2h^2} = \frac{1}{2} r^2 \dot{\theta}^2 - \frac{\mu}{r}$$

When,

$e < 1 \quad then \quad E_T < 0 \qquad -\frac{\mu}{r}\left(\frac{Potential\ Energy}{mass}\right) > \frac{1}{2}V^2\left(\frac{Kinetic\ Energy}{mass}\right)$, the trajectory is an ELLIPSE (Kepler's first law)

$e = 1 \quad then \quad E_T = 0 \qquad -\frac{\mu}{r}\left(\frac{Potential\ Energy}{mass}\right) = \frac{1}{2}V^2\left(\frac{Kinetic\ Energy}{mass}\right)$, the trajectory is a PARABOLA

$e > 1 \quad then \quad E_T > 0 \qquad -\frac{\mu}{r}\left(\frac{Potential\ Energy}{mass}\right) < \frac{1}{2}V^2\left(\frac{Kinetic\ Energy}{mass}\right)$, the trajectory is a HYPERBOLA

Further Reading:

1. Mechanics & Thermodynamics of Propulsion (2nd Ed.) by Philip G. Hill and Carl Peterson (2nd Edition) Chapter #10 (section 10.5)

2. Introduction to Space Flight by Francis J. Hale (Chapter #3) Prentice Hall.

3. Space Flight Dynamics William E Wiesel (2nd Edition) Chapter #3.